

Unit-06

Quantum Theory of hydrogen atom

自我練習題目與例題

Example 6.4

A sample of a certain element is placed in a 0.300-T magnetic field and suitably excited. How far apart are the Zeeman components of the 450-nm spectral line of this element?

Solution

The separation of the Zeeman components is

$$\Delta\nu = \frac{eB}{4\pi m}$$

Since $\nu = c/\lambda$, $d\nu = -c d\lambda/\lambda^2$, and so, disregarding the minus sign,

$$\begin{aligned}\Delta\lambda &= \frac{\lambda^2 \Delta\nu}{c} = \frac{eB\lambda^2}{4\pi mc} \\ &= \frac{(1.60 \times 10^{-19} \text{ C})(0.300 \text{ T})(4.50 \times 10^{-7} \text{ m})^2}{(4\pi)(9.11 \times 10^{-31} \text{ kg})(3.00 \times 10^8 \text{ m/s})} \\ &= 2.83 \times 10^{-12} \text{ m} = 0.00283 \text{ nm}\end{aligned}$$

31. Find the minimum magnetic field needed for the Zeeman effect to be observed in a spectral line of 400-nm wavelength when a spectrometer whose resolution is 0.010 nm is used.

6-31: See Example 6.4; solving for B ,

$$\begin{aligned} B &= \frac{\Delta\lambda}{\lambda^2} \frac{4\pi m c}{e} \\ &= \frac{(0.010 \times 10^{-9} \text{ m})}{(400 \times 10^{-9} \text{ m})^2} \frac{4\pi (9.1095 \times 10^{-31} \text{ kg}) (2.998 \times 10^8 \text{ m/s})}{(1.602 \times 10^{-19} \text{ C})} = 1.34 \text{ T}. \end{aligned}$$

32. The Zeeman components of a 500-nm spectral line are 0.0116 nm apart when the magnetic field is 1.00 T. Find the ratio e/m for the electron from these data.

6-32: See Example 6.4; solving for the ratio e/m in terms of B , $\Delta\lambda$ and λ ,

$$\frac{e}{m} = \frac{4\pi c \Delta\lambda}{B \lambda^2} = \frac{4\pi (2.998 \times 10^8 \text{ m/s}) (0.0116 \times 10^{-9} \text{ m})}{(1.00 \text{ T}) (500 \times 10^{-9} \text{ m})^2} = 1.75 \times 10^{11} \text{ C/kg}.$$

The actual ratio, using values from the front endpapers, is

$$\frac{e}{m} = \frac{1.602 \times 10^{-19} \text{ C}}{9.1095 \times 10^{-31} \text{ kg}} = 1.759 \times 10^{11} \text{ C/kg}.$$

24. A hydrogen atom is in the $4p$ state. To what state or states can it go by radiating a photon in an allowed transition?

6-24: From Equation (6.36), the final state must be an s or d state, and the possible final states are $3s$, $3d$, $2s$ (there is no $2d$ state) or $1s$ (there is no $1d$ state, either).

27. Verify that the $n = 3 \rightarrow n = 1$ transition for the particle in a box of Sec. 5.8 is forbidden whereas the $n = 3 \rightarrow n = 2$ and $n = 2 \rightarrow n = 1$ transitions are allowed.

6-27: The relevant integrals are of the form

$$\int_0^L x \sin \frac{n \pi x}{L} \sin \frac{m \pi x}{L} dx.$$

The integrals may be found in a number of ways, including consulting tables or using symbolic-manipulation programs (see, for instance, the solution to Problem 5-15 for sample Maple commands that are easily adapted to this problem).

One way to find a general form for the integral is to use the identity

$$\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

and the indefinite integral (found from integration by parts)

$$\int x \cos kx \, dx = \frac{x \sin kx}{k} - \frac{1}{k} \int \sin kx \, dx = \frac{x \sin kx}{k} + \frac{\cos kx}{k^2}$$

to find the above definite integral as

$$\frac{1}{2} \left[\begin{aligned} & \frac{Lx}{(n-m)\pi} \sin \frac{(n-m)\pi x}{L} + \frac{L^2}{(n-m)^2 \pi^2} \cos \frac{(n-m)\pi x}{L} \\ & - \frac{Lx}{(n+m)\pi} \sin \frac{(n+m)\pi x}{L} + \frac{L^2}{(n+m)^2 \pi^2} \cos \frac{(n+m)\pi x}{L} \end{aligned} \right]_0^L,$$

where $n^2 \neq m^2$ is assumed. The terms involving sines vanish, with the result of

$$\frac{L^2}{2\pi^2} \left[\frac{\cos(n-m)\pi - 1}{(n-m)^2} - \frac{\cos(n+m)\pi - 1}{(n+m)^2} \right].$$

If n and m are both odd or both even, $n + m$ and $n - m$ are even, the arguments of the cosine terms in the above expression are even-integral multiples of π , and the integral vanishes. Thus, the $n = 3 \rightarrow n = 1$ transition is forbidden, while the $n = 3 \rightarrow n = 2$ and $n = 2 \rightarrow n = 1$ transitions are allowed.

To make use of symmetry arguments, consider that

$$\int_0^L \left(x - \frac{L}{2}\right) \sin \frac{n \pi x}{L} \sin \frac{m \pi x}{L} dx = \int_0^L x \sin \frac{n \pi x}{L} \sin \frac{m \pi x}{L} dx$$

for $n \neq m$, because the integral of L times the product of the wave functions is zero; the wave functions were shown to be orthogonal in Chapter 5 (again, see Problem 5-15). Letting $u = \frac{L}{2} - x$,

$$\sin \frac{n \pi x}{L} = \sin \frac{n \pi ((L/2) - u)}{L} = \sin \left(\frac{n \pi}{2} - \frac{n \pi u}{L} \right).$$

This expression will be $\pm \cos\left(\frac{n\pi u}{L}\right)$ for n odd and $\pm \sin\left(\frac{n\pi u}{L}\right)$ for n even. The integrand is then an odd function of u when n and m are both even or both odd, and hence the integral is zero. If one of n or m is even and the other odd, the integrand is an even function of u and the integral is nonzero.

28. In the Bohr model of the hydrogen atom, what is the magnitude of the orbital magnetic moment of an electron in the n th energy level?

6-28: The magnitude of the angular momentum of an electron in the n^{th} energy level is $L = n \hbar$, and from Equation (6.39), the magnitude of the orbital magnetic moment of such an electron is

$$\mu = \left(\frac{e}{2m} \right) L = n \frac{\hbar e}{2m}.$$

29. Show that the magnetic moment of an electron in a Bohr orbit of radius r_n is proportional to $\sqrt{r_n}$.

6-29: From Equation (6.39), the magnitude of the magnetic moment of an electron in a Bohr orbit is proportional to the magnitude of the angular momentum, and hence proportional to n . The orbital radius is proportional to n^2 (see Equation (4.13) or Problem 4-28), and so the magnetic moment is proportional to $\sqrt{r_n}$.

Evaluation of Integrals for Hydrogen Wave Functions

Many of the problems in this chapter involve integrals of the form

$$\int u^n e^{-u} du \quad \text{or} \quad \int_0^\infty u^n e^{-u} du$$

for n a nonnegative integer.

Such integrals are well-known and may be found from tables or by use of symbolic-manipulation programs. Standard methods of finding these integrals involve repeated integration by parts. Specifically, for $n = 0$, the indefinite integral is readily found,

$$\int e^{-u} du = -e^{-u}$$

and the improper definite integral is seen to be

$$\int_0^\infty e^{-u} du = 1.$$

Integrating the above indefinite integral by parts,

$$\begin{aligned} \int e^{-u} du &= u e^{-u} - \int u d(-e^{-u}) \\ &= u e^{-u} + \int u e^{-u} du, \end{aligned}$$

so that

$$\int u e^{-u} du = -e^{-u} (u + 1).$$

The improper definite integral is seen to be

$$\int_0^{\infty} u e^{-u} du = 1.$$

The process may be repeated, or the process of integration by parts may be generalized as

$$\int u^n e^{-u} du = u^{n+1} e^{-u} du + n \int u^{n+1} e^{-u} du.$$

The first few such integrals are:

$$n = 0 \quad \int e^{-u} du = -e^{-u}$$

$$n = 1 \quad \int u e^{-u} du = -e^{-u} (u + 1)$$

$$n = 2 \quad \int u^2 e^{-u} du = -e^{-u} (u^2 + 2u + 2)$$

$$n = 3 \quad \int u^3 e^{-u} du = -e^{-u} (u^3 + 3u^2 + 6u + 6)$$

$$n = 4 \quad \int u^4 e^{-u} du = -e^{-u} (u^4 + 4u^3 + 12u^2 + 24u + 24)$$

Further integrals may be found by, for instance, variations on the Maple command

```
>for m from 0 to 10 do int(u^m*exp(-u),u) od;
```

For the definite improper integrals, a pattern, easily verified by induction is found, the standard result

$$\int_0^{\infty} u^n e^{-u} du = n!.$$

The Maple command that would demonstrate this pattern is

```
>for n from 0 to 10 do int(u^n*exp(-u),u=0..infinity) od;
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