

Unit-05

Quantum Mechanics

自我練習題目與例題

3. Which of the following wave functions cannot be solutions of Schrödinger's equation for all values of x ? Why not? (a) $\psi = A \sec x$; (b) $\psi = A \tan x$; (c) $\psi = Ae^{x^2}$; (d) $\psi = Ae^{-x^2}$.

5-3: The functions (a) and (b) are both infinite when $\cos x = 0$, at $x = \pm\pi/2, \pm3\pi/2, \dots, \pm(2n+1)\pi/2$ for any integer n , and neither $\psi = A \sec x$ or $\psi = A \tan x$ could be a solution of Schrödinger's equation for all values of x . The function (c) diverges as $x \rightarrow \pm\infty$, and cannot be a solution of Schrödinger's equation for all values of x .

4. Find the value of the normalization constant A for the wave function $\psi = Axe^{-x^2/2}$.

5-4: From $\psi^*\psi = A^2 e^{-x^2}$,

$$A^2 \int_{-\infty}^{\infty} e^{-x^2} dx = A^2 \sqrt{\pi} = 1, \quad \text{so}$$

$$A^2 = \sqrt{\frac{4}{\pi}} \quad \text{and} \quad A = \left(\frac{4}{\pi}\right)^{1/4}.$$

(See the end of this chapter, or the discussion preceding the solution to Problem 5-26, for a possible method of determining the integral.)

5. The wave function of a certain particle is $\psi = A \cos^2 x$ for $-\pi/2 < x < \pi/2$. (a) Find the value of A . (b) Find the probability that the particle be found between $x = 0$ and $x = \pi/4$.

5-5: Both parts involve the integral $\int \cos^4 x \, dx$, evaluated between different limits for the two parts. Of the many ways to find this integral, including consulting tables and using symbolic-manipulation programs, a direct algebraic reduction gives

$$\begin{aligned}\cos^4 x &= (\cos^2 x)^2 = \left[\frac{1}{2}(1 + \cos 2x) \right]^2 \\ &= \frac{1}{4} [1 + 2 \cos 2x + \cos^2(2x)] \\ &= \frac{1}{4} \left[1 + 2 \cos 2x + \frac{1}{2}(1 + \cos 4x) \right] \\ &= \frac{3}{8} + \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x,\end{aligned}$$

where the identity $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$ has been used twice.

(a) The needed normalization condition is

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} \psi^* \psi dx &= A^2 \int_{-\pi/2}^{\pi/2} \cos^4 x dx \\ &= A^2 \left[\frac{3}{8} \int_{-\pi/2}^{\pi/2} dx + \frac{1}{2} \int_{-\pi/2}^{\pi/2} \cos 2x dx + \frac{1}{8} \int_{-\pi/2}^{\pi/2} \cos 4x dx \right] = 1.\end{aligned}$$

The integrals

$$\int_{-\pi/2}^{\pi/2} \cos 2x dx = \frac{1}{2} \sin 2x \Big|_{-\pi/2}^{\pi/2} \quad \text{and} \quad \int_{-\pi/2}^{\pi/2} \cos 4x dx = \frac{1}{4} \sin 4x \Big|_{-\pi/2}^{\pi/2}$$

are seen to vanish, and the normalization condition reduces to

$$1 = A^2 \left(\frac{3}{8} \right) \pi, \quad \text{or} \quad A = \sqrt{\frac{8}{3\pi}}.$$

(b) Evaluating the same integral between the different limits,

$$\int_0^{\pi/4} \cos^4 x dx = \left[\frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x \right]_0^{\pi/4} = \frac{3\pi}{32} + \frac{1}{4}.$$

The probability of the particle being found between $x = 0$ and $x = \pi/4$ is the product of this integral and A^2 , or

$$A^2 \left(\frac{3\pi}{32} + \frac{1}{4} \right) = \frac{8}{3\pi} \left(\frac{3\pi}{32} + \frac{1}{4} \right) = 0.462.$$

6. The formula $y = A \cos \omega(t - x/v)$, as we saw in Sec. 3.3, describes a wave that moves in the $+x$ direction along a stretched string. Show that this formula is a solution of the wave equation, Eq.(5.3).

5-6: This is quite similar to the situation of Example 5.1; in fact, some might recognize $A \cos \omega(t - x/v)$ as the real part of $A e^{-i\omega(t - x/v)}$. Using the same technique as Example 5.1, with $u = u(x, t) = \omega(x - t/v)$,

$$\frac{d}{du} \cos u = -\sin u, \quad \frac{d}{du} (-\sin u) = -\cos u, \quad \text{and so}$$

$$\begin{aligned} \frac{\partial y}{\partial x} &= -A \sin \omega(t - x/v) \left(-\frac{\omega}{v}\right), & \frac{\partial^2 y}{\partial^2 x} &= -A \cos \omega(t - x/v) \left(-\frac{\omega}{v}\right)^2 = -\frac{\omega^2}{v^2} y, \\ \frac{\partial y}{\partial t} &= -A \cos \omega(t - x/v) (\omega), & \frac{\partial^2 y}{\partial^2 t} &= -A \cos \omega(t - x/v) (-\omega)^2 = -\omega^2 y, \end{aligned}$$

and so Equation (5.3) is satisfied.

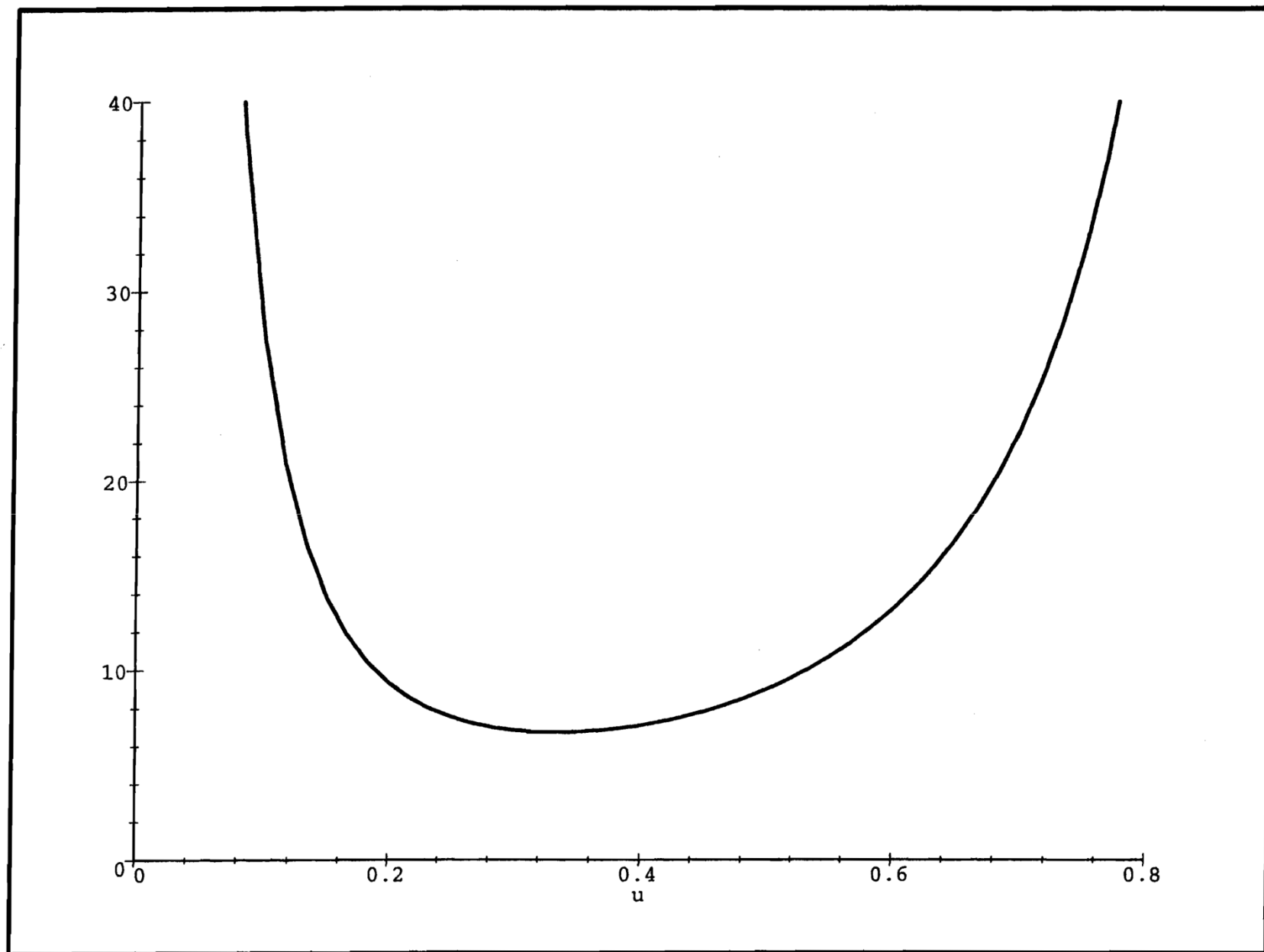
16. A rigid-walled box that extends from $-L$ to L is divided into three sections by rigid interior walls at $-x$ and x , where $x < L$. Each section contains one particle in its ground state. (a) What is the total energy of the system as a function of x ? (b) Sketch $E(x)$ versus x . (c) At what value of x is $E(x)$ a minimum?

5-16: The total energy will be the sum of three terms, each corresponding to one of the particles, and each in the form of Equation (5.40), with $n = 1$ and the widths of the regions $L - x$, $L - x$ and $2x$.

(a) The total energy is then

$$E_T = \frac{\pi^2 \hbar^2}{2m} \left[\frac{2}{(L-x)^2} + \frac{1}{4x^2} \right] = \frac{\pi^2 \hbar^2}{2m L^2} \left[\frac{2}{(1-(x/L))^2} + \frac{1}{4(x/L)^2} \right].$$

(b) The total energy E_T , in multiples of $\frac{\pi^2 \hbar^2}{2m L^2}$, as a function of $u = x/L$, is shown:



18. As noted in Exercise 8, a linear combination of two wave functions for the same system is also a valid wave function. Find the normalization constant B for the combination

$$\psi = B \left(\sin \frac{\pi x}{L} + \sin \frac{2\pi x}{L} \right)$$

of the wave functions for the $n = 1$ and $n = 2$ states of a particle in a box L wide.

5-18: The normalization condition is

$$\begin{aligned} \int_0^L \psi^* \psi dx &= 1 = B^2 \int_0^L \left(\sin \frac{\pi x}{L} + \sin \frac{2\pi x}{L} \right)^2 dx \\ &= B^2 \left[\int_0^L \sin^2 \frac{\pi x}{L} dx + 2 \int_0^L \sin \frac{\pi x}{L} \sin \frac{2\pi x}{L} dx + \int_0^L \sin^2 \frac{2\pi x}{L} dx \right]. \end{aligned}$$

At this point, the integrals may be evaluated directly by standard methods, or it can be noted that the first and last integrals are both $L/2$ (see Equation (4.43)), and from Problem 5-15, the middle term is 0. The normalization condition is then

$$1 = B^2 L \quad \text{so} \quad B = \frac{1}{\sqrt{L}}.$$

20. In Sec. 3.7 the standard deviation σ of a set of N measurements of some quantity x was defined as

$$\sigma = \sqrt{\frac{1}{N} \sum_{i=1}^N (x_i - x_0)^2}$$

- (a) Show that, in terms of expectation values, this formula can be written as

$$\sigma = \sqrt{\langle (x - \langle x \rangle)^2 \rangle} = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

- (b) If the uncertainty in position of a particle in a box is taken as the standard deviation, find the uncertainty in the expectation value $\langle x \rangle = L/2$ for $n = 1$. (c) What is the limit of Δx as n increases?

5-20: In the notation used, both in this problem and in Section 3.7, x_0 is the *mean* of the measurements,

$$x_0 = \langle x \rangle = \frac{1}{N} \sum_{i=1}^N x_i.$$

(a) For algebraic simplicity, consider the *variance*,

$$V(x_i) = \sigma^2 = \frac{1}{N} \sum_{i=1}^N (x_i - x_0)^2 = \frac{1}{N} \sum_{i=1}^N x_i^2 - \frac{2}{N} \sum_{i=1}^N x_i x_0 + \frac{1}{N} \sum_{i=1}^N x_0^2.$$

In the expression on the right in the above equation:

$$\frac{1}{N} \sum_{i=1}^N x_i^2 \quad \text{is} \quad \langle x^2 \rangle,$$

$$\frac{2}{N} \sum_{i=1}^N x_i x_0 = 2 x_0 \frac{1}{N} \sum_{i=1}^N x_i = 2 x_0^2 = 2 \langle x \rangle^2$$

$$\frac{1}{N} \sum_{i=1}^N x_0^2 = x_0^2 \frac{1}{N} \sum_{i=1}^N 1 = x_0^2 = \langle x \rangle^2,$$

so that $V(x_i) = \sigma^2 = \langle x^2 \rangle - 2 \langle x \rangle^2 + \langle x \rangle^2 = \langle x^2 \rangle - \langle x \rangle^2.$

(b) Using the result of Problem 5-17, for a particle confined to a box of length L , with $n = 1$,

$$\langle x^2 \rangle = L^2 \left(\frac{1}{3} - \frac{1}{2\pi^2} \right).$$

From Example 5.5, $\langle x \rangle = L/2$, so

$$V = \sigma^2 = \langle x^2 \rangle_1 - \langle x \rangle^2 = L^2 \left(\frac{1}{3} - \frac{1}{2\pi^2} \right) - \frac{L^2}{4} = L^2 \left(\frac{1}{12} - \frac{1}{2\pi^2} \right).$$

Numerically, this is $\sigma = L(0.181)$.

(c) For arbitrary n , again using the result of Problem 5-17,

$$V_n = \sigma_n^2 = \langle x^2 \rangle_n - \langle x \rangle^2 = L^2 \left(\frac{1}{3} - \frac{1}{2n\pi^2} \right) - \frac{L^2}{4} = L^2 \left(\frac{1}{12} - \frac{1}{2n\pi^2} \right).$$

As n increases, σ_n increases, and as $n \rightarrow \infty$, $\sigma_n \rightarrow L/\sqrt{12} = L(0.289)$.

21. A particle is in a cubic box with infinitely hard walls whose edges are L long (Fig. 5.18). The wave functions of the particle are given by

$$\psi = A \sin \frac{n_x \pi x}{L} \sin \frac{n_y \pi y}{L} \sin \frac{n_z \pi z}{L} \quad \begin{array}{l} n_x = 1, 2, 3, \dots \\ n_y = 1, 2, 3, \dots \\ n_z = 1, 2, 3, \dots \end{array}$$

Find the value of the normalization constant A .

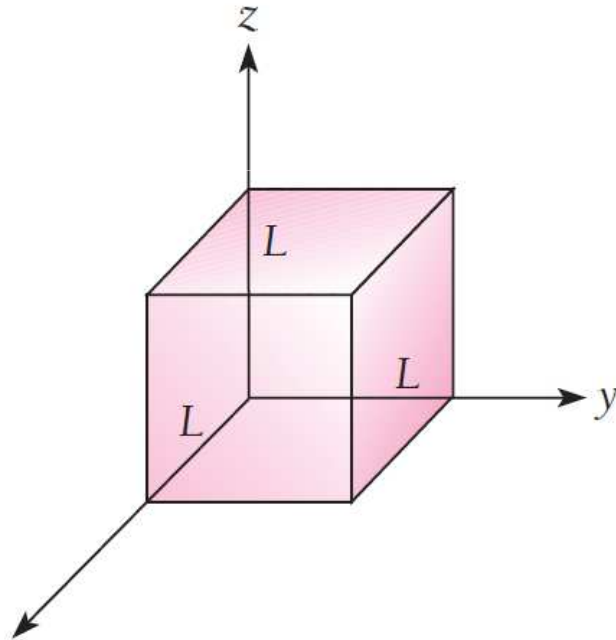


Figure 5.18 A cubic box.

5-21: The normalization constant, assuming A to be real, is given by

$$\begin{aligned}\int \psi^* \psi dV &= 1 = \int \psi^* \psi dx dy dz \\ &= A^2 \left(\int_0^L \sin^2 \frac{n_x \pi x}{L} dx \right) \left(\int_0^L \sin^2 \frac{n_y \pi y}{L} dy \right) \left(\int_0^L \sin^2 \frac{n_z \pi z}{L} dz \right).\end{aligned}$$

Each integral above is equal to $\frac{L}{2}$ (from calculations identical to Equation (5.43)).

The result is

$$A^2 \left(\frac{L}{2} \right)^3 = 1 \quad \text{or} \quad A = \left(\frac{2}{L} \right)^{3/2}.$$

24. Electrons with energies of 0.400 eV are incident on a barrier 3.00 eV high and 0.100 nm wide. Find the approximate probability for these electrons to penetrate the barrier.

5-24: Using the approximate form for the transmission probability as given in Equation (5.60),

$$\begin{aligned} T &\approx e^{(-2 k_2 L)} = e^{\left(-2 \sqrt{\frac{2 m (U - E)}{\hbar}} L \right)} \\ &= e^{\left(-2 \frac{\sqrt{2 (9.1095 \times 10^{-31} \text{ kg}) (1.602 \times 10^{-19} \text{ J/eV})}}{(1.055 \times 10^{-34} \text{ J}\cdot\text{s})} (0.100 \times 10^{-9} \text{ m}) \right)} \\ &= 0.19. \end{aligned}$$

25. A beam of electrons is incident on a barrier 6.00 eV high and 0.200 nm wide. Use Eq. (5.60) to find the energy they should have if 1.00 percent of them are to get through the barrier.

5-25: Solving Equation (5.60) for k_2 ,

$$k_2 = \frac{1}{2L} \ln \frac{1}{T} = \frac{1}{2(0.200 \times 10^{-9} \text{ m})} \ln(100) = 1.1513 \times 10^{10} \text{ m}^{-1},$$

keeping extra significant figures. Equation (5.86), from the Appendix, may be solved for the energy E , but a more direct expression is

$$\begin{aligned} E &= U - \text{KE} = U - \frac{p^2}{2m} = U - \frac{(\hbar k_2)^2}{2m} \\ &= 6.00 \text{ eV} - \frac{\left((1.055 \times 10^{-34} \text{ J}\cdot\text{s}) (1.1513 \times 10^{10} \text{ m}^{-1})\right)^2}{2(9.1095 \times 10^{-31} \text{ kg})(1.602 \times 10^{-19} \text{ J/eV})} \\ &= 0.949 \text{ eV}. \end{aligned}$$