

Problem 1

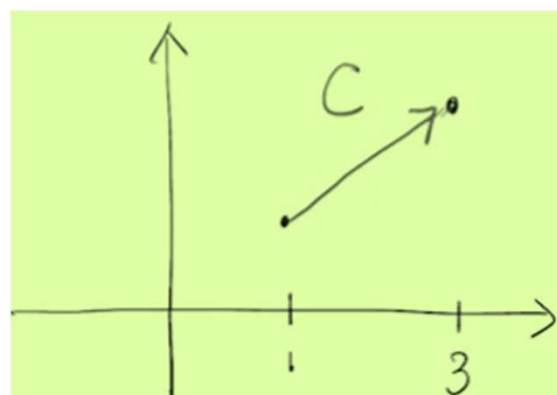
AM1-Q05B

- 在一維空間中有一個力量與座標的關係為 $f(x) = kx^2$ ，請計算當這個物體在一維線上從坐標 $x = 2$ 移動至 $x = 4$ ，過程中這個力量所做的功。
- 在二維空間中有一個作用力向量，這個力量的數學形式如下：

$$\vec{F} = x^2 \hat{j}.$$

有一個物體在二維空間中從坐標 $(1, 1)$ 沿著直線位移到坐標 $(3, 3)$ ，請計算此過程中這個力量做的功。

$$W = \int_C \vec{F} \cdot d\vec{s} = \int_2^4 kx^2 dx \quad 64-8$$
$$= \frac{1}{3} kx^3 \Big|_2^4 = \frac{56}{3} k$$



$$x = 1 + t, dx = dt$$
$$y = 1 + t, dy = dt$$
$$d\vec{s} = dx \hat{i} + dy \hat{j}$$
$$= dt \hat{i} + dt \hat{j}$$

$$\vec{F} = x^2 \hat{j} = (1+t)^2 \hat{j}$$

$$\vec{F} \cdot d\vec{s} = (1+t)^2 dt$$

$$W = \int_C \vec{F} \cdot d\vec{s} = \int_0^2 (1+2t+t^2) dt$$

$$= \left[t + t^2 + \frac{1}{3}t^3 \right]_0^2 = 26/3$$

Problem 2
一個向量場

AM1-Q05B

$$\vec{a} = y\hat{i} + x\hat{j}$$

· 計算下列圖中的路徑積分 C_1, C_2, C_3, C_4 。

(a) $I_1 = \int_{C_1} \vec{a} \cdot d\vec{s}$.

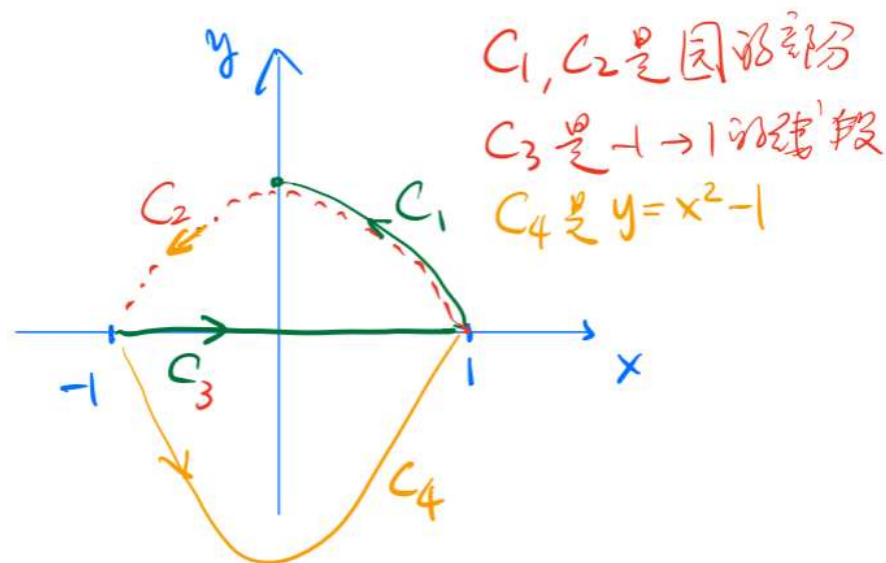
(b) $I_2 = \int_{C_2} \vec{a} \cdot d\vec{s}$.

(c) $I_3 = \int_{C_3} \vec{a} \cdot d\vec{s}$.

(d) $I_4 = \int_{C_4} \vec{a} \cdot d\vec{s}$.

你可以驗證向量場
 $\vec{a}$ 是一個保守場

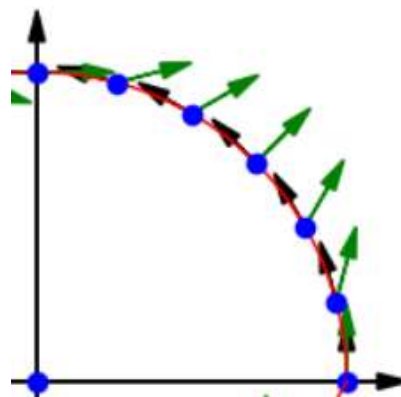
C_1, C_2 是單位圓 $x^2 + y^2 = 1$ 的部分，其中前者涵蓋第一象限的部分，後者是涵蓋一二象限的半圓。 C_3 是 -1 到 1 沿著正 x 軸方向的線段。 C_4 是拋物線 $y = x^2 - 1$ 的一部分，介於 -1 與 1 之間。線積分的路徑方向如圖中的箭頭顯示。



$$C_1: \int_0^{\frac{\pi}{2}} (\sin\theta \hat{i} + \cos\theta \hat{j}) \cdot (-\sin\theta \hat{i} + \cos\theta \hat{j}) d\theta = \frac{1}{2} \sin 2\theta \Big|_0^{\frac{\pi}{2}} = 0$$

$$= \int_0^{\frac{\pi}{2}} (-\sin^2\theta + \cos^2\theta) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos 2\theta d\theta$$



$$\int_0^{\frac{\pi}{4}} = \frac{1}{2}$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} = -\frac{1}{2}$$

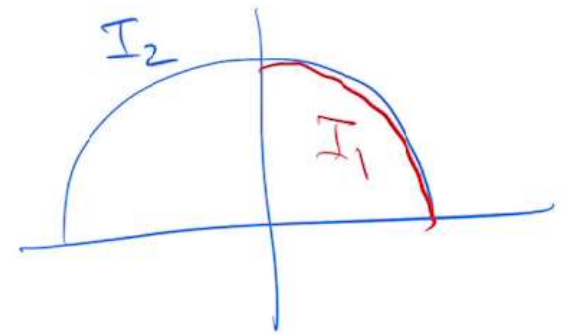
$$C_1 = 0$$

$$C_2 = 0$$

$$C_3 = 0$$

$$C_4 = 0$$

$$I_2 = \int_{C_2} \vec{a} \cdot d\vec{s} = 2 \int_{C_1} \vec{a} \cdot d\vec{s} = 2I_1 = 0$$

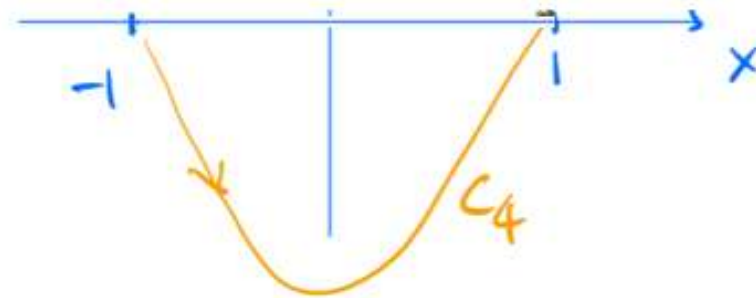


$$I_3 = \int 0 dx = 0, \quad y=0 \text{ along } x\text{-axis}$$



$$\text{for } C_4: y = x^2 - 1, \quad dy = 2x dx$$

$$\begin{aligned} \vec{a} \cdot d\vec{s} &= (y, x) \cdot (dx, 2x dx) \\ &= y dx + 2x^2 dx \\ &= (x^2 - 1) dx + 2x^2 dx \\ &= 3x^2 dx - dx \end{aligned}$$



$$\int_{-1}^1 3x^2 dx - \int_{-1}^1 dx = x^3 \Big|_{-1}^1 - 2 = 2 - 2 = 0$$

$$C_1 = 0$$

$$C_2 = 0$$

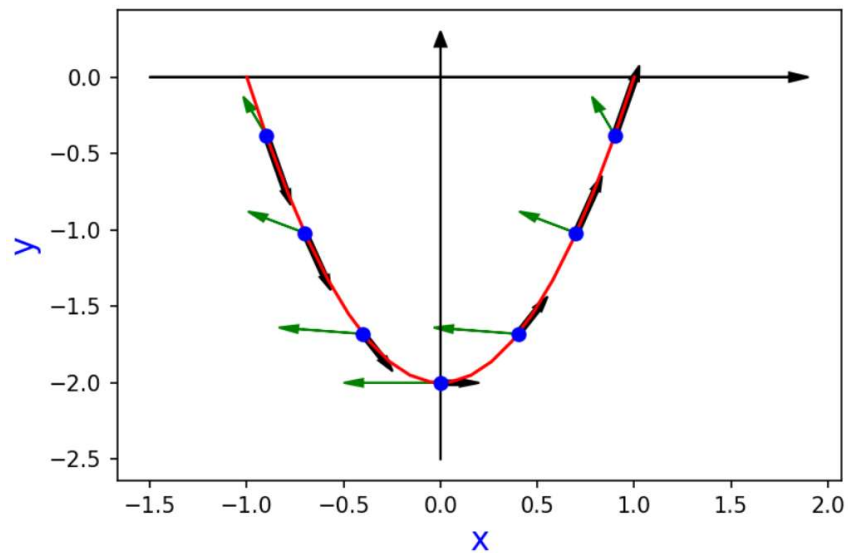
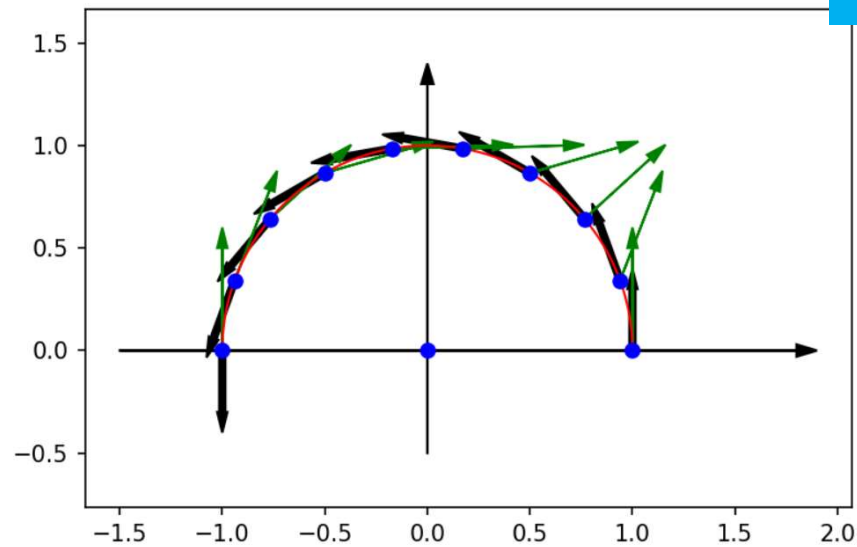
$$C_3 = 0$$

$$C_4 = 0$$

AM1-Q05B

$$\vec{a} = y\hat{i} + x\hat{j}$$

前面一題
稍加變化



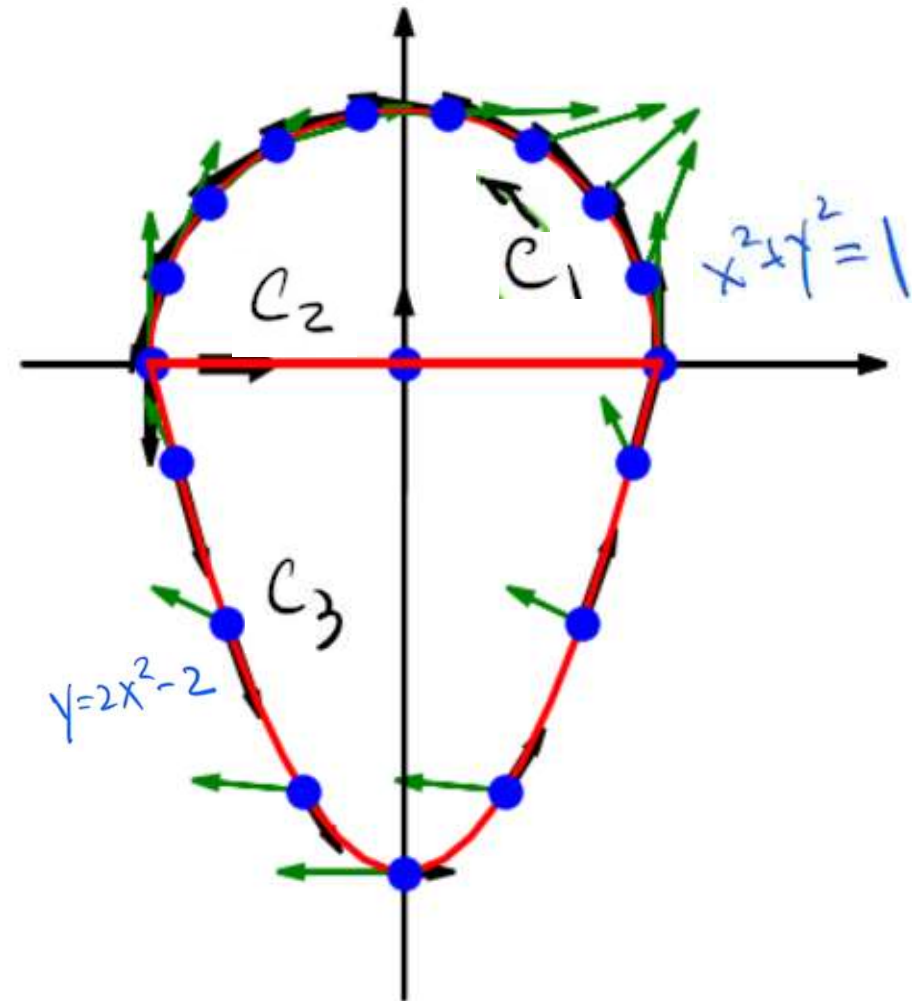
$$\vec{a} = (y, x^2)$$

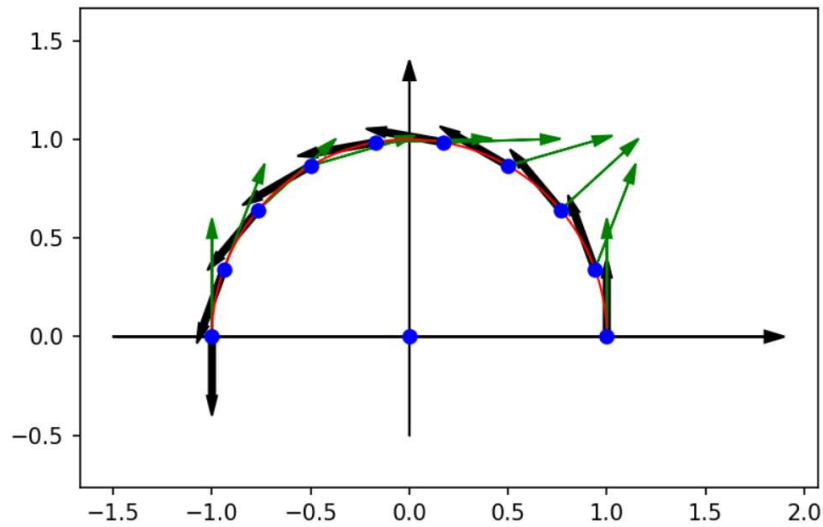
P Q

$$\frac{\partial P}{\partial y} = 1$$

$$\frac{\partial Q}{\partial x} = 2x$$

$\vec{a}$ is not conservative





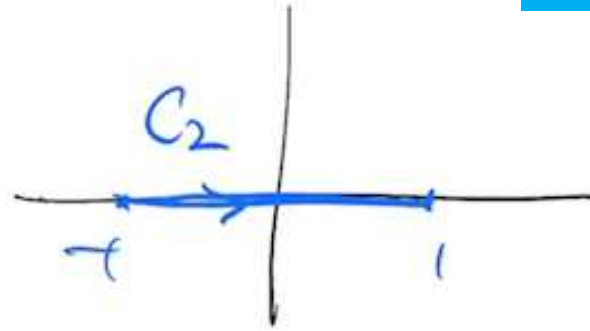
$$C_1: x = \cos\theta, y = \sin\theta, d\vec{s} = (-\sin\theta, \cos\theta)d\theta$$

$$\vec{a} \cdot d\vec{s} = (-\sin^2\theta + \cos^3\theta)d\theta$$

$$I_1 = \int_{C_1} \vec{a} \cdot d\vec{s} = \int_0^\pi (-\sin^2\theta + \cos^3\theta) d\theta$$

$$= -\int_0^\pi \frac{1}{2}(1 - \cos 2\theta) d\theta + \int_0^\pi (1 - \sin^2\theta) d\sin\theta$$

$$= -\frac{\pi}{2} + \sin\theta \Big|_0^\pi - \frac{1}{3} \sin^3\theta \Big|_0^\pi = -\frac{\pi}{2}$$



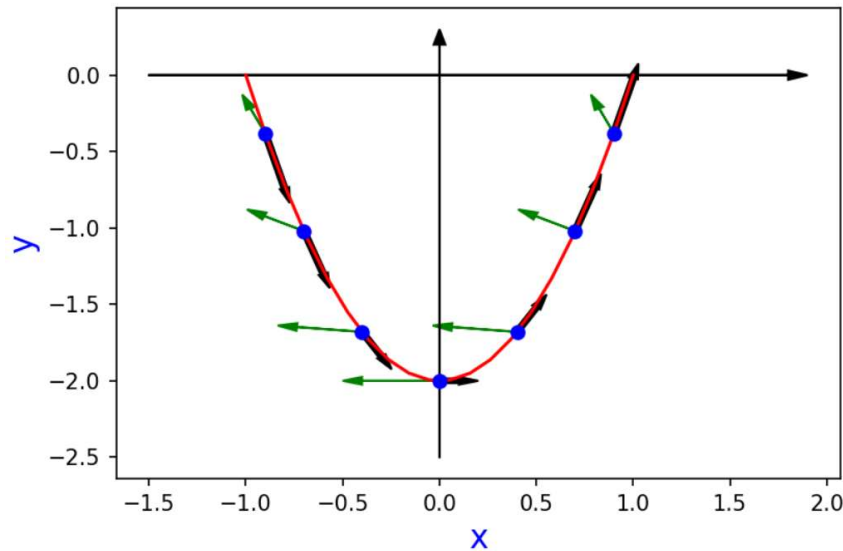
$$C_2: d\vec{s} = dx \hat{i}, \quad \vec{a} = y \hat{i} + x^2 \hat{j}$$

$$y = 2x^2 - 2$$

$$\int y dx = \int_{-1}^1 (2x^2 - 2) dx$$

$$= \left. \frac{2}{3} x^3 \right|_{-1}^1 - \left. 2x \right|_{-1}^1$$

$$= \frac{4}{3} - 4 = -\frac{8}{3}$$



$$C_3: y = 2x^2 - 2$$

$$\frac{dy}{dx} = 4x, \quad dy = 4x dx$$

$$\vec{a} \cdot d\vec{s} = (y, x^2) (dx, 4x dx) \\ = (2x^2 - 2) dx + 4x^3 dx$$

$$I_3 = \int_{-1}^1 (4x^3 + 2x^2 - 2) dx = \frac{2}{3} (1+1) - 4 \\ = \frac{4}{3} - 4 = -\frac{8}{3} \approx -2.667$$

AM1-Q05B

```
import matplotlib
import matplotlib.pyplot as plt
import numpy as np
matplotlib.use("Agg")
```

```
t=np.linspace(0,1*np.pi,100)
dt=t[1]-t[0]
x=np.cos(t)
y=np.sin(t)
tx=-np.sin(t);
ty=np.cos(t)
ax=np.sin(t)
ay=np.cos(t)**2
Linteg_1=sum(ax*tx+ay*ty)*dt
print('Linteg_1=',Linteg_1)
```

```
plt.axis("equal")
hl=0.1
plt.plot(x,y,'r-', lw='1')
plt.plot(0.0,0.0, 'bo')
plt.plot(1.2,1.4, 'wo', ms='1')
t=np.linspace(0,np.pi,10)
xc=np.cos(t)
yc=np.sin(t)
tx=-np.sin(t); ty=np.cos(t)
plt.arrow(-1.5,0,3.4-
hl,0,head_width=0.02*3,
head_length=hl*1, fc='k', ec='k' )
plt.arrow(0,-0.5,0,1.9-
hl,head_width=0.02*3, head_length=hl*1,
fc='k', ec='k' )
plt.plot(xc,yc,'bo')
sa=0.5
for i in range(10):
```

```
txi=tx[i]*0.3
tyi=ty[i]*0.3
axi=yc[i]; ayi=xc[i]**2; ax=axi*sa;
ay=ayi*sa
```

```
plt.arrow(xc[i],yc[i],txi,tyi,width=0.03,head
_width=0.05, head_length=hl, fc='k', ec='k' )
plt.arrow(xc[i],yc[i],ax,ay,head_width=0.05, head_length=hl, fc='g', ec='g' )
plt.savefig('line-integ1.png',dpi=150)
```

```
plt.figure()
NP2=101
x=np.linspace(-1,1,NP2)
dx=x[1]-x[0]
y=2*x**2-2
tx=np.array([dx for i in range(NP2)])
ty=4*x*dx
ax=y
ay=x**2
Linteg_2=sum(ax*tx+ay*ty)
print('Linteg_2=',Linteg_2)
```

```
xp=np.linspace(-1,1,20)
yp=2*xp**2-2
plt.plot(xp,yp,'r-')
xp=np.array([-0.9,-0.7,-0.4,0,0.4,0.7,0.9])
yp=2*xp**2-2
plt.plot(xp,yp,'bo')
for i in range(len(xp)):
txi=dx*5
tyi=4*xp[i]*dx*5
```

```
ax=0.2*yp[i]
ay=0.2*xp[i]**2
#print(i,xp[i],yp[i],txi,tyi)
```

```
plt.arrow(xp[i],yp[i],txi,tyi,width=0.03,head
_width=0.05, head_length=hl, fc='k', ec='k' )
plt.arrow(xp[i],yp[i],ax,ay,head_width=0.05,
head_length=hl, fc='g', ec='g' )
```

```
plt.arrow(-1.5,0,3.4-
hl,0,head_width=0.02*3,
head_length=hl*1, fc='k', ec='k' )
plt.arrow(0,-2.5,0,2.8-
hl,head_width=0.02*3, head_length=hl*1,
fc='k', ec='k' )
plt.xlabel('x', fontsize="15", color='b')
plt.ylabel('y', fontsize="15", color='b')
#plt.show()
plt.savefig('line-integ2.png',dpi=150)
print ('plot is done')
```

https://physexp.thu.edu.tw/~AP/YC/AM1/HTML/Line_integ_1.py

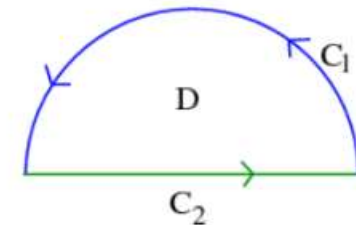
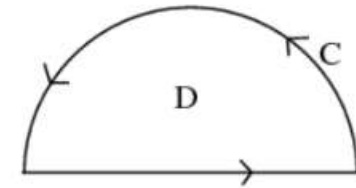
這一個python程式提供了這個題目的畫圖和數值線積分。

Problem 3

Consider a line integration $\oint_C y^2 dx + 3xy dy$ along a curve of half circle as shown in the figure. According to Green's theorem : $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3y - 2y = y$

$$\begin{aligned}\oint_C y^2 dx + 3xy dy &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\ &= \iint_D y dA\end{aligned}$$

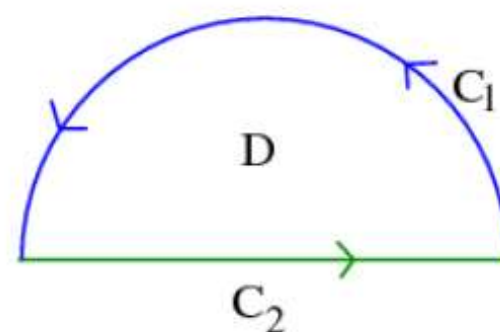
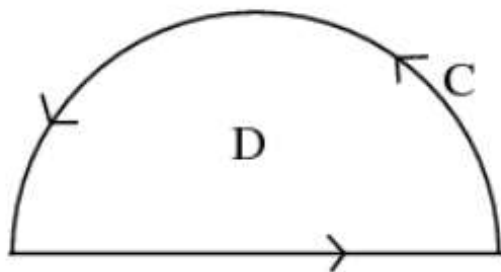
The line integration is equal to a double integration of an area enclosed by the curve C . Calculate both integrations and verify the theorem. (01小題)



$$\mathbf{c}(t) = (\cos t, \sin t), 0 \leq t \leq \pi.$$

$$\mathbf{c}'(t) = (-\sin t, \cos t).$$

$$\begin{aligned}& \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\ &= \iint_D y dA \\ &= \iint_D y dA \\ &= \int_{-1}^1 \int_0^{\sqrt{1-x^2}} y dy dx \\ &= \int_{-1}^1 \left(\frac{y^2}{2} \Big|_{y=0}^{y=\sqrt{1-x^2}} \right) dx \\ &= \int_{-1}^1 \frac{1-x^2}{2} dx \\ &= \frac{x}{2} - \frac{x^3}{6} \Big|_{-1}^1 = \frac{2}{3}.\end{aligned}$$



$$\mathbf{c}(t) = (\cos t, \sin t), 0 \leq t \leq \pi.$$

$$\mathbf{c}'(t) = (-\sin t, \cos t).$$

$$\oint_C y^2 dx + 3xy dy$$

$$= \int_{\pi}^0 \vec{F} \cdot d\vec{s}$$

$$= \int_0^{\pi} F(\cos t, \sin t) \cdot (-\sin t, \cos t) dt$$

$$= \int_0^{\pi} (\sin^2 t, 3 \cos t \sin t) \cdot (-\sin t, \cos t) dt$$

$$= \int_0^{\pi} (-\sin^3 t + 3 \sin t \cos^2 t) dt$$

$$= \int_0^{\pi} (-\sin t(1 - \cos^2 t) + 3 \sin t \cos^2 t) dt$$

$$= \int_0^{\pi} (-\sin t + 4 \sin t \cos^2 t) dt$$

$$\int_0^{\pi} \sin t dt = 2$$

$$\begin{aligned} \int_0^{\pi} \sin t \cos^2 t dt &= \int_1^{-1} -u^2 du \\ &= -\frac{u^3}{3} \Big|_1^{-1} = -(-1)/3 + 1/3 = 2/3. \end{aligned}$$

$$\int_{C_2} \vec{F} \cdot d\vec{s} = -2 + 4(2/3) = -2 + 8/3 = 2/3.$$

$$\int_{C_1} \vec{F} \cdot d\vec{s} = 0.$$

$$\int_C \vec{F} \cdot d\vec{s} = \int_{C_1} \vec{F} \cdot d\vec{s} + \int_{C_2} \vec{F} \cdot d\vec{s} = \frac{2}{3} + 0 = \frac{2}{3}.$$

Problem 4

Evaluate $I = \int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y, z) = xz\vec{i} - yz\vec{k}$ and C is the curve given by

$$\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}, \quad 0 \leq t \leq 1. \text{ (01小題)}$$

$I =$ _____

04: ANS:=3

Solution:

$$\begin{aligned} \vec{r}(t) &= (1-t)\langle -1, 2, 0 \rangle + t\langle 3, 0, 1 \rangle \\ &= \langle 4t-1, 2-2t, t \rangle, \quad 0 \leq t \leq 1 \end{aligned}$$

$$\begin{aligned} \vec{F}(\vec{r}(t)) &= (4t-1)(t)\vec{i} - (2-2t)(t)\vec{k} \\ &= (4t^2-t)\vec{i} - (2t-2t^2)\vec{k} \end{aligned}$$

$$\vec{r}'(t) = \langle 4, -2, 1 \rangle$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = 4(4t^2-t) - (2t-2t^2) = 18t^2 - 6t$$

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_0^1 (18t^2 - 6t) dt \\ &= (6t^3 - 3t^2) \Big|_0^1 \\ &= 3 \end{aligned}$$

Problem 5

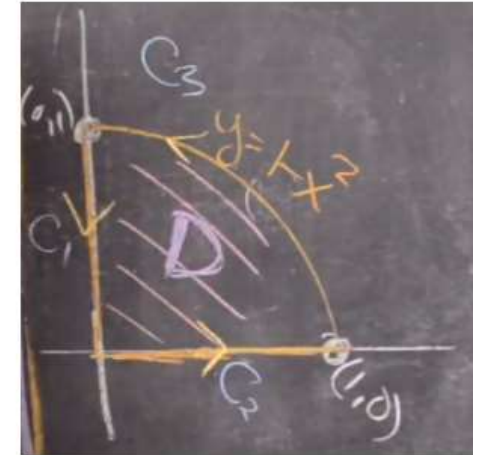
Verify the Green's theorem for the following integrations: (a) line integration:

$$\oint_C xdx + ydy$$

C is the curve consist with line segments from $(0,1)$ to $(0,0)$ and $(0,0)$ to $(1,0)$ and a part of parabola $y = 1 - x^2$ from $(1,0)$ to $(0,1)$ as shown in the figure. (b) Double integration:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

with P and Q from the integrand of the line integral above. (01小題)



這個問題的解答請參考下面這個影片：

https://www.youtube.com/watch?v=_wToO33okAI

End of AM1-Q05B