

Problem 1

Evaluate the line integral $I = \int [(x^2 - y^2)dx - 2xydy]$ along the following paths from $(0, 0)$ to $(1, 1)$. (05小題)

along the curve $y = x^2$, $I = \underline{\hspace{2cm}}$

01: ANS:=-0.667

along the curve $x = t^2$ and $y = t$, $I = \underline{\hspace{2cm}}$

02: ANS:=-0.667

the integration can be rewritten as $I = \int \vec{a} \cdot d\vec{r}$.

$\vec{a} = a_x(x, y)\hat{i} + a_y(x, y)\hat{j}$. (02小題)

$a_x = \underline{\hspace{2cm}} [x, y]$

03: ANS:=-x**2-y**2

$a_y = \underline{\hspace{2cm}}$

04: ANS:=-2*x*y

calculate $A = |\nabla \times \vec{a}|$ for the point $(1, 2, 3)$.

$A(1, 2, 3) = \underline{\hspace{2cm}}$

05: ANS:=0

$$(a) y = x^2 \Rightarrow dy = 2x dx$$

$$I = \int [(x^2 - (x^2)^2)dx - 2x(x^2)(2x dx)]$$

$$= \int_0^1 (x^2 - x^4 - 4x^4) dx$$

$$= \int_0^1 (x^2 - 5x^4) dx$$

$$= \frac{1}{3}x^3 \Big|_0^1 - x^5 \Big|_0^1 = \frac{1}{3} - 1 = -\frac{2}{3}$$

$$(b) x = t^2, y = t$$

$$dx = 2t dt, dy = dt$$

$$I = \int_0^1 [(t^4 - t^2)(2t dt) - 2(t^2)(t)dt]$$

$$= \int_0^1 (2t^5 - 2t^3 - 2t^3) dt$$

$$= \frac{2}{6} - \frac{4}{4} = \frac{1}{3} - 1 = -\frac{2}{3}$$

$$\vec{a} = (x^2 - y^2, -2xy, 0)$$

$$\vec{\nabla} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 - y^2 & -2xy & 0 \end{vmatrix}$$

$$= 0\hat{i} + 0\hat{j} + (-2y + 2y)\hat{k} \\ = \vec{0}$$

Problem 2

1. 在物理學當中我們利用線積分來定義做功，請寫下功的線積分定義。

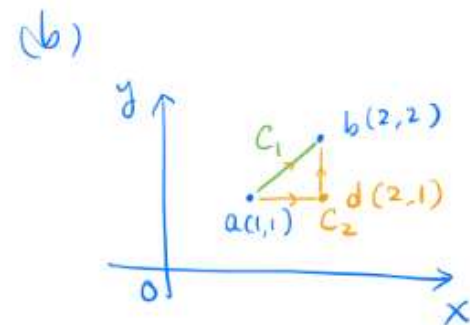
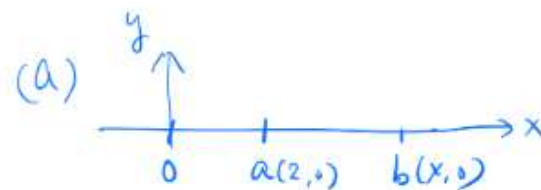
2. 請利用上面的定義，計算彈性力所做的功。這是一個一維的問題，路徑就是直線

$a(2,0) \rightarrow b(x,0)$, $x > 2$ 。滿足虎克定律的彈性力數學形式為 $f = -kx$ 。

3. 其次我們來計算在二維空間中重力 $\vec{f} = -mg\hat{j}$ 所做的功。假設初始點為 $a(1,1)$ 點，終點是 $b(2,2)$

點。我們可以計算兩個路徑所做的功。如圖所示 C_1 是連接 a, b 兩點的直線。

4. C_2 由兩條直線組成，其中經過 $d(1,2)$ 點。



$$\begin{aligned}
 W &= \int_a^b \vec{f} \cdot d\vec{s} \\
 &= \int_2^x (-kx) dx = -k\left(\frac{1}{2}x^2\right)_2^x \\
 &= \frac{1}{2}(4)k - \frac{1}{2}kx^2
 \end{aligned}$$

$$\begin{aligned}
 W &= \int_a^b (-mg\hat{j}) \cdot (dx\hat{i} + dy\hat{j}) \\
 &= \int_a^b -mg dy = -mgy \Big|_1^2 = -mg \cdot 1
 \end{aligned}$$

Problem 3

Evaluate the line integral $\oint_c \vec{G} \cdot d\vec{r}$ where c is the unit circle, and in Cartesian coordinates, $\vec{G} = y^3 \hat{i} - x^3 \hat{j}$. (01小題)

$$\vec{G} = y^3 \hat{i} - x^3 \hat{j} = P(x,y) \hat{i} + Q(x,y) \hat{j}$$

Use Green's Thm :

$$\oint_c \vec{G} \cdot d\vec{r} = \oint_c P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

$$= \iint_R (-3x^2 - 3y^2) dx dy$$

$$= -3 \iint_R (x^2 + y^2) dx dy$$

$$= -3 \int_0^{2\pi} d\theta \int_0^1 r^2 \cdot r dr = -6\pi \cdot \frac{1}{4} = -\frac{3\pi}{2}$$

08: **ANS:=-4.712**

直接做封閉路徑的線積分，也可以得到相同的積分結果。

$$\begin{aligned}\oint \vec{G} \cdot d\vec{r} &= \oint y^3 dx - x^3 dy \\ &\begin{array}{l} \swarrow \\ X = r \cos \theta, \quad dx = -r \sin \theta d\theta \\ Y = r \sin \theta, \quad dy = r \cos \theta d\theta \end{array} \\ &= \int_0^{2\pi} \sin^3 \theta (-r \sin \theta d\theta) - r \cos^3 \theta r \cos \theta d\theta \\ &= -\int_0^{2\pi} (\sin^4 \theta + r^4 \cos^4 \theta) d\theta \\ &= -\int_0^{2\pi} \{ (\sin^2 \theta + r^2 \cos^2 \theta)^2 - 2 \sin^2 \theta r^2 \cos^2 \theta \} d\theta \\ &= -2\pi + \frac{1}{2} \int_0^{2\pi} (\sin 2\theta)^2 d\theta \\ &= -2\pi + \frac{1}{2} \cdot \frac{1}{2} \cdot 2\pi \\ &= -\frac{3}{2} \pi\end{aligned}$$

Problem 4

Calculate the line integral(I) of the vector field $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ along the following two paths joining the origin to the point $P(1, 1, 1)$. (i) Along a straight line joining the origin to P, (ii) along a path parameterized by $x = t, y = t^2, z = t^3$. (02/小題)

(i) Along a straight line joining the origin to P, $I = \underline{\hspace{2cm}}$

16: ANS:=1.333

(ii) along a path parameterized by $x = t, y = t^2, z = t^3$,
 $I = \underline{\hspace{2cm}}$

17: ANS:=0.842

(a) Along the straight line path, we have, $x=y=z$, so that $\int \vec{F} \cdot d\vec{l} = \int (F_x dx + F_y dy + F_z dz)$ can be

expressed as $\int_0^1 4x^2 dx - \int_0^1 y^2 dy + \int_0^1 z^2 dz = \frac{4}{3}$.

(b) Substituting the parameterized form, we have, $dx = dt, dy = 2t dt, dz = 3t^2 dt$. The line integral is

$$\int_0^1 4t^4 dt - \int_0^1 2t^5 dt + \int_0^1 3t^7 dt = \frac{101}{120}$$

Problem 5

AM1-Q05

Calculate the line integral of $\vec{F} = (x^2 + y^2)\hat{i} + xy\hat{j}$ over a quarter circle in the upper half plane along the path connecting (3,0) to (0,3). (01小題)

The path integral is most conveniently done by parameterizing $x = 3 \cos \theta, y = 3 \sin \theta$, so that $dx = -3 \sin \theta, dy = 3 \cos \theta$. The integral is then given by

$$-27 \int_0^{\frac{\pi}{2}} \sin \theta d\theta + 27 \int_0^{\frac{\pi}{2}} \cos^2 \theta \sin \theta d\theta = -18. \text{ If}$$

the curve is traversed in the opposite direction, the line integral would become +18.

line integral=_____

18: **ANS:=-18**

Problem 6

請完成下面的二重積分，被積分的函數

$f(x, y) = x^2 + y^2$ ，積分的區域有兩個分別是 R_1 正方形區域， R_2 為 $1/4$ 圓的面。

$$\int \int_{R_1} f(x, y) \, dx dy = ?$$

$$\int \int_{R_2} f(x, y) \, dx dy = ?$$

(01小題)

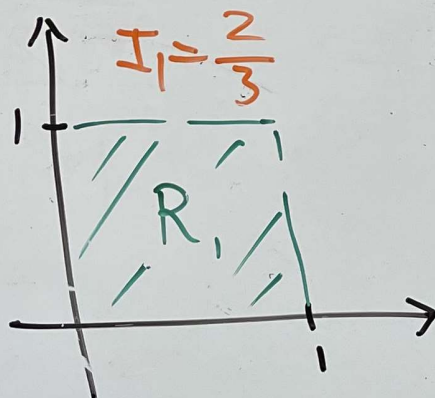


Problem 6

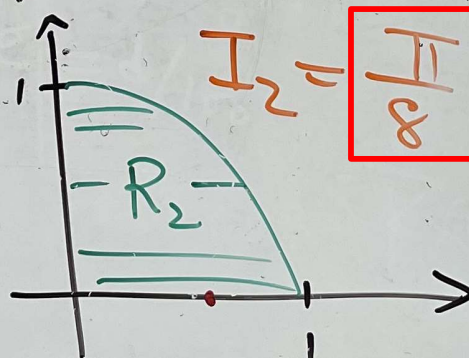
$$\iint_R f(x,y) dx dy$$

$$f(x,y) = x^2 + y^2$$

R_1 :



R_2 :



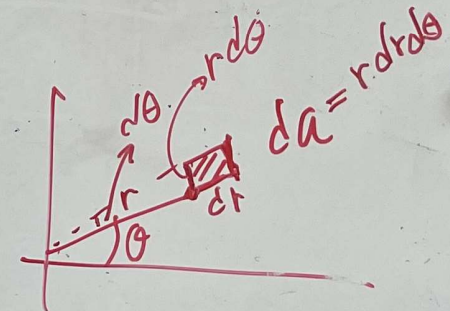
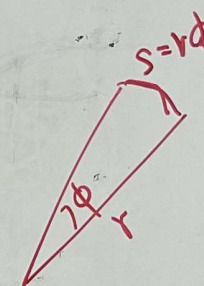
$$\int_0^1 \int_0^1 (x^2 + y^2) dy dx = \int_0^1 \left(x^2 + \frac{1}{3} \right) dx = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$\int_0^1 \int_{\pi/4}^{\pi/2} r^2 (r dr d\theta) = \int_0^1 r^3 dr \int_{\pi/4}^{\pi/2} d\theta$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$da = dx dy = r dr d\theta$$



Problem 7

請完

成下面的線積分。(01小題)

12 Ans. = 嘗試 = 1 分數 = 10

$$(1) \int_C (x^2 + y^2) ds = \underline{\hspace{2cm}}$$



$$\int_C \vec{a} \cdot d\vec{s}$$

$$(2) \vec{a} = x\hat{i} + y\hat{j}$$

$$(3) \vec{a} = -y\hat{i} + x\hat{j}$$

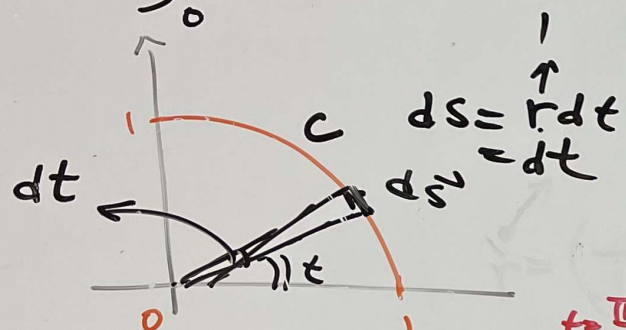
$$(4) \vec{a} = -4\hat{j}$$

$$(5) \vec{a} = x^2\hat{i}$$

Problem 7

$$(1) \int_C (x^2 + y^2) ds = \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} 1 dt = \frac{\pi}{2}$$



$$x(t) = \cos t$$

$$y(t) = \sin t$$

$$\frac{1}{3} u^3 \Big|_{t=0}^{t=\frac{\pi}{2}} = \frac{1}{3} u^3 \Big|_0^1 = \frac{1}{3} (1^3 - 0^3) = \frac{1}{3}$$

$$\int_{\frac{\pi}{2}}^0 x y ds$$

$$du = \cos t$$

$$du = -\sin t dt$$

$$\int_0^{\frac{\pi}{2}} \cos t \sin t dt = -\int_1^0 u du = -\frac{1}{2} u^2 \Big|_1^0 = \frac{1}{2}$$

$$\int_C \vec{a} \cdot d\vec{s}$$

$$(2) \vec{a}_1 = x \hat{i} + y \hat{j}$$

$$I_1 = 0$$

$$(3) \vec{a}_2 = -y \hat{i} + x \hat{j}$$

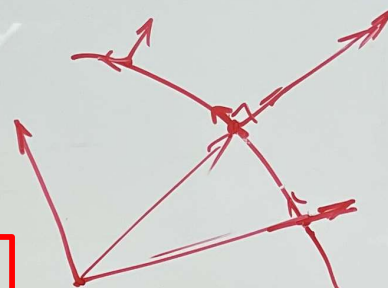
$$I_2 = \frac{\pi}{2} \quad ds = dt$$

$$(4) \vec{a}_3 = -4 \hat{j}$$

$$(5) \vec{a}_4 = x^2 \hat{i}$$

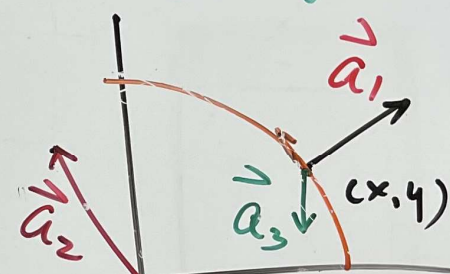
$$I_4 = \int_C \vec{a}_4 \cdot d\vec{s} = \int_0^{\frac{\pi}{2}} (\cos^2 t)(\cos t) dt$$

$$= -4 \sin t \Big|_0^{\frac{\pi}{2}} = -4 = I_3$$



$$I_1 = 0$$

$$d\vec{s} = -\sin t \hat{i} + \cos t \hat{j}$$



$$\vec{a}_3 \cdot d\vec{s} = -4 \cos t$$

$$I_3 = \int_0^{\frac{\pi}{2}} -4 \cos t dt$$

End of AM1-Q05